Revision of proofs for MEG

Common methods of proof we studied:

- Induction
- Direct
- Contradiction
- Contrapositive

A **conditional statement** has the form:

If P is true, then Q is true. This can be written as $P \Rightarrow Q$. We assume P is true and that Q follows. P is called the "hypothesis" and Q is called the "conclusion".

In a **direct proof** for a conditional statement you assume P is true, and show that Q follows. $P \Rightarrow Q$.

The **converse** of $P \Rightarrow Q$ is $Q \Rightarrow P$.

If the converse equals the statement, we call P and Q **equivalent**. In logical notation: $P \Rightarrow Q$ and $Q \Rightarrow P$ then $P \Leftrightarrow Q$. Even more notation: $(P \Rightarrow Q) \cap (Q \Rightarrow P) \Rightarrow (P \Leftrightarrow Q)$.

The **contrapositive** of $P \Rightarrow Q$ is $Q' \Rightarrow P'$. Here, the ' symbol means "not". Sometimes proving the contrapositive is easier than giving a direct proof. This <u>isn't a "contradiction"</u> since we're assuming something is true and then proving, not assuming something is true and disproving it. The contrapositive of the contrapositive is of course the original statement.

Proof by **contradiction** is when you assume the negation ("opposite") of what you're trying to solve, and by disproving that assumption you prove that the original (non-negation) must be true.

Mathematical **induction** is a method used to prove that a statement is true over the integers.

The most common mathematical formulation of induction, over the natural numbers:

Let $S(n)$ be a statement involving n. If $S(1)$ holds and for every $k \geq 1, S(k) \Rightarrow S(k+1)$, then for every $n \geq 1$, the statement $S(n)$ holds.

Induction can be extended over the natural numbers, to include negative integers:

Let $S(n)$ denote a statement regarding an integer n, and let $k \in \mathbb{Z}$ be fixed. If:

$S(k)$ holds

For every $m \geq k, S(m) \Rightarrow S(m+1)$

Then for every $n \geq k$ the statement $S(n)$ holds. (If we work backwards from k instead of forwards, i.e. $m \leq k, S(m) \Rightarrow S(m-1)$ then we can prove on a larger domain and up to all of the integers, from $(-\infty, +\infty)$.

Here, k is acting as our **base case** (this is a significant feature of induction proofs). In a textbook, most proofs with induction will be done with $k = 0$ or 1.

Quiz:

<u>Direct Proof 1:</u>

Let x and y be any two positive real numbers.

Prove that

$$\frac{x+y}{2} \geq \sqrt{xy}$$

<u>Direct Proof 2:</u>

For $n \in \mathbb{Z}$, prove that $5n^2 + 3n + 7$ is odd.

<u>Proof by Contrapositive 1:</u>

Let $n, m \in \mathbb{Z}$ and consider the statement: If mn and m+n are even, then m and n are even. Prove the contrapositive. You will have to consider <u>cases</u>.

<u>Proof by Contrapositive 2:</u>

Let x and y be positive real numbers, $x, y \in \mathbb{R}^+$.

Prove that:

$$\sqrt{x} - \sqrt{y} = \frac{x-y}{\sqrt{x}+\sqrt{y}}$$

Hence prove that if $x > y$, then $\sqrt{x} > \sqrt{y}$.

Give a simpler proof by considering the contrapositive.

<u>Proof by Contradiction 1:</u>

Logarithmic irrationality: Suppose for prime $p \neq 2$, $p^x = 2$. Prove that x is irrational.

<u>Proof by Contradiction 2:</u>

Irrationality of roots: Let p be a prime number. Show that $\sqrt{p}$ is also irrational.

<u>Proof by Induction 1:</u>

AM-GM inequality using Cauchy induction:

For the positive reals $a_1, a_2, \dots a_n$,

$$\frac{(a_1 + a_2 + \cdots + a_n)}{n} \geq \sqrt[n]{a_1 a_2 \dots a_n}$$

<u>Proof by Induction 2:</u>

Proof of binomial theorem:

Prove that:

$$(x+a)^n = \sum_{k=0}^{n} \binom{n}{k} x^k a^{n-k}$$

Where:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

And $n!$ within the positive integers is defined as:

$$n! = 1 \times 2 \times \ldots \times (n-1) \times n = \prod_{i=1}^{n} i$$

Proof by Induction 3:

Prove that:

$$e^x > x + 1, \quad x \in \mathbb{Z}^+$$

Challenge: Prove for $x \in \mathbb{R}^+$ ($x \in (0, \infty)$)